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Coefficient Rings of Multidimensional Torus Extensions

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INTRODUCTION

In this paper commutative rings with unity and unital homomorphisms will be considered. By X, Y we always mean torsion-free abelian groups, with a subscript specifying the rank, if necessary. We intend to investigate homomorphisms between group rings of such groups. We apply the notation and standard results from [6, 12].

We will often make use of homomorphisms $\varphi\colon AX \to BX$ with an additional property: $\varphi(x) = x$ for all $x \in X$. We agree to call them *X-homomorphisms.*

As usual, we call a ring A *X-invariant* if for any ring B we have $A \approx B$ whenever the group rings AX, BX are isomorphic. One can also introduce two other types of X-invariance which are more effective in a sense. A ring A is *strongly X-invariant* provided for any ring B and isomorphism $\varphi\colon AX \to BX$ there exists a B-automorphism $\sigma\colon BX \to BX$ such that $\sigma\varphi$ is an X-homomorphism. Also, a ring A is *totally X-invariant* if for any ring B and isomorphism $\varphi\colon AX \to BX$ we have $\varphi(A) = B$. The both types of invariance have been considered for group rings or polynomial rings in [1, 4, 7, 13], although not necessarily under the same names.

It is easy to see that any totally X-invariant ring is strongly X-invariant, and also that all strongly X-invariant rings are just X-invariant. One can observe that A is strongly X-invariant if and only if for any isomorphism $\varphi\colon AX \to BX$ the set $\varphi(X)$ is a B-module basis for BX.

Our goal is to prove that all results concerning the three types of invariance which are valid for the infinite cyclic group remain true for an arbitrary free abelian group of finite rank. Group rings of such groups can be interpreted in algebraic geometry as multidimensional torus extensions. In particular the invariance properties can be studied in this context. Our results show that one can reduce to the case of one-dimensional torus.

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In Section 4 and 5 we demonstrate that further generalization of our results about free abelian groups to arbitrary torsion free groups is, in general, impossible.

1. Preliminaries

We start with a brief review of some notions and results. We recall that each idempotent of AX lies necessarily in A. This leads to the following

PROPOSITION 1.1. *Let e be a nontrivial idempotent of A. Then*

(a) *A is strongly X-invariant iff eA, $(1-e)A$ are strongly X-invariant*

(b) *A is totally X-invariant iff eA, $(1-e)A$ are totally X-invariant.*

Proof of Case (a). Let A be strongly X-invariant and let $\varphi\colon (eA)X \to BX$ be an isomorphism. Set $C = B \times (1-e)A$. Now φ together with id: $(1-e)\,AX \to (1-e)\,AX$ give rise to an isomorphism $\bar{\varphi}\colon AX \to CX$. By assumption, we have to our disposal a C-automorphism $\bar{\sigma}\colon CX \to CX$ such that $\bar{\sigma}\bar{\varphi}$ is an X-homomorphism. Call σ the restriction of $\bar{\sigma}$ to BX. Now $\sigma\varphi$ is an X-homomorphism, i.e., eA is strongly X-invariant. Substituting $1-e$ for e we obtain that $(1-e)A$ is strongly X-invariant as well.

Now assume that eA and $(1-e)A$ are strongly X-invariant. Let $\varphi\colon AX \to BX$ be an isomorphism. Write $f = \varphi(e)$. As $f \in B$ so

$$BX \cong fBX \oplus (1-f)\,BX.$$

Obviously φ respects this decomposition and so it can be written as $\varphi = \varphi_1 \times \varphi_2$ for some isomorphisms $\varphi_1\colon eAX \to fBX$, $\varphi_2\colon (1-e)\,AX \to (1-f)\,BX$. By assumption, we have the corresponding automorphisms σ_1 and σ_2 such that $\sigma_1 \circ \varphi_1$, $\sigma_2 \circ \varphi_2$ are X-homomorphisms. Finally, set

$$\sigma(x) = \sigma_1(fx) + \sigma_2((1-f)x) \qquad \text{for} \quad x \in X.$$

The B-automorphism σ defined in this way, when composed with φ, gives an X-automorphism. This completes the proof of case (a). The proof of case (b) is similar. ∎

COROLLARY 1.2. *Let $A = A_1 \oplus \cdots \oplus A_r$. The ring A is strongly (totally) X-invariant iff each A_i is strongly (totally) X-invariant.*

If $P(R)$ denotes the prime radical (nilradical) of R then $P(AX) = (P(A))X$. Moreover, each minimal prime ideal of AX is of the form IX for some minimal prime ideal I in A.

We also write $U(R)$ for the group of invertible elements in R. Let

$E=\{e_1,\ldots, e_n\}$ be a decomposition of the unity element in a ring A, i.e., a decomposition into a sum of orthogonal idempotents. If $u \in U(AX)$ then we say that *E splits u* if there exist $v \in U(A)$, $x_1,\ldots, x_n \in X$ (not necessarily different) and $p \in P(AX)$ such that

$$u = v \cdot \sum e_i x_i + p.$$

For brevity, we agree to call any such presentation of u *standard*. It is proved in [11] (see also [7, 9]) that standard presentations exist and are, to some extend, unique. In particular we call $u \in U(AX)$ *elementary* if it is splitable by $E=\{1\}$. Such units forms a subgroup in $U(AX)$, denoted further by $U_{el}(AX)$.

Like in [10] we call a homomorphism φ: $AX \to BY$ *elementary* if $\varphi(x) \subseteq U_{el}(BY)$. When X and Y are of finite rank such homomorphisms constitute the building blocks for all homomorphisms $AX \to BY$.

LEMMA 1.3. *Let X, Y be groups of finite rank and let φ: $AX \to BY$ be an isomorphism. Then there exist ideals $A_i \subseteq A$, $B_i \subseteq B$ for $1 \leqslant i \leqslant n$, such that*

$$A = A_1 \oplus \cdots \oplus A_n, \qquad B = B_1 \oplus \cdots \oplus B_n,$$

and

$$\varphi|_{A_iX}: \quad A_iX \to B_iY$$

as well as

$$\varphi^{-1}|_{B_iY}: \quad B_iY \to A_iX$$

are elementary homomorphisms.

Proof. One follows the same path as in the proof of Lemma 2.1 in [10] and additionally uses the fact that $u^k \in U_{el}(AX)$ implies $u \in U_{el}(AX)$ (cf. [9, 11]). It is known that the rank assumption above is necessary even if we restrict to A-isomorphisms; see [9]. ∎

Now, let φ: $AX \to BY$ be an elementary homomorphism. Then for any $x \in X$ there exists a unique $y \in Y$ such that $\varphi(x) = vy + p$ where $v \in U(B)$ and $p \in P(BY)$. We denote this element y by $h_\varphi(x)$. It is easy to see that $h_\varphi: X \to Y$ defined above is a group homomorphism. The role of this function in the study of homomorphisms between group rings is illustrated by the following result, proved in fact by Lantz in [11] (see also [9]).

THEOREM 1.4. *Let φ be an elementary A-endomorphism of AX. Then*

(a) *if φ is an automorphism then h_φ is an automorphism,*

(b) *if h_φ is an automorphism and either A is reduced or X is finitely generated then φ is an automorphism.* ∎

COROLLARY 1.5. *Let A be a reduced ring and let $\varphi\colon AX \to AY$ be an elementary A-isomorphism. Then X is isomorphic to Y.*

Proof. We notice that h_φ is the required isomorphism. ▮

2. CHANGE OF RANK THEOREMS

In this section we always assume that X is a free abelian group of finite rank. We start from a characterisation of rings which are strongly X-invariant.

THEOREM 2.1. *A ring A is strongly X-invariant iff for any ring B and isomorphism $\varphi\colon AX \to BX$ holds $\varphi(\Delta(A)) \subseteq U(B) + P(BX)$.*

Proof. Let A be a strongly X-invariant ring and $\varphi\colon AX \cong BX$. As the group $U(B) + P(BX)$ is preserved by B-automorphisms so without loss of generality we can assume that φ is an X-homomorphism. Take $u \in U(A)$ and consider the standard presentation of $\varphi(u)$:

$$\varphi(u) = v \cdot \sum e_i x_i + p.$$

Assume that some x_i, say x_1, is different from 1. Write $e = e_1$ and $f = \varphi^{-1}(e)$. Then φ induces an X-isomorphism $fAX \to eBX$ such that

$$\varphi(fu) = vex_1 + p' \qquad \text{with} \quad p' \in P(eBX).$$

Pick a basis $\{y_1, \ldots, y_r\}$ for X so that $x_1 = y_1^k$ for some $k > 0$. Consider the A-endomorphism τ of AX given by

$$\tau(y_1) = fuy_1 \qquad \text{and} \qquad \tau(y_i) = y_i \text{ for } i > 1.$$

Then $\varphi\tau$ is an elementary isomorphism $fAX \to eBX$. From Proposition 1.1. We conclude that fA is strongly X-invariant. Thus there exists an eB-automorphism $\sigma\colon eBX \to eBX$ such that $\sigma\varphi\tau$ is an X-homomorphism. Therefore $\sigma^{-1}(x) = \varphi\tau(x)$ and in particular:

$$\sigma^{-1}(y_1) = \varphi(fuy_1) = evy_1^{k+1} + p'',$$
$$\sigma^{-1}(y_i) = y_i \qquad \text{for} \quad i > 1,$$

where $p'' \in P(eBX)$. From this it easily follows that σ^{-1} is an elementary automorphism of eBX, but $h_{\sigma^{-1}}$ is not an automorphism of X. This contradicts Theorem 1.4. In this way we have shown that

$$\varphi(u) = v + p \qquad \text{with} \quad v \in U(B),\ p \in P(BX).$$

For the converse, let A be a ring such that for all rings B and for all isomorphisms $\varphi: AX \to BX$ holds

$$\varphi(U(A)) \subseteq U(B) + P(BX).$$

Fix one such φ. In view of Lemma 1.3 and Corollary 1.2 we can assume that φ and φ^{-1} are elementary. Thus h_φ is an endomorphism of X. Let $x \in X$. We can write $\varphi^{-1}(x) = uy + p$ with $u \in U(A)$, $y \in X$, and $p \in P(AX)$. In particular $x = \varphi(u) \cdot \varphi(y) + \varphi(p)$. By assumption, $\varphi(u) = v_1 + p_1$ with $v_1 \in U(B)$, $p_1 \in P(BX)$. Because φ is elementary, we also get $\varphi(y) = v_2 x_2 + p_2$ where $v_2 \in U(B)$, $x_2 \in X$, $p_2 \in P(BX)$. Putting all these together we obtain

$$x = v_1 v_2 x_2 + q \qquad \text{for some} \quad q \in P(BX).$$

This implies $x_2 = x$, thus $h_\varphi(y) = x$ and so h_φ is a surjection. Because X is finitely generated, so h_φ is an automorphism.

Let τ be the B-endomorphism of BX given by $\tau(x) = \varphi(x)$ for $x \in X$. Obviously τ is elementary and $h_\tau = h_\varphi$ is an automorphism of X. Again, by Theorem 1.4, we obtain that τ is an automorphism. Setting $\sigma = \tau^{-1}$ we get an X-isomorphism $\varphi \circ \sigma$, and so A is strongly X-invariant. ▮

COROLLARY 2.2. *A ring A is totally X-invariant iff for any ring B and isomorphism $\varphi: AX \to BX$ holds $\varphi(A) \subseteq B$.*

Proof. By Theorem 2.1 we can assume that φ is an X-isomorphism. But then $BX = \varphi(A)X$.

Corollary 2.2 can also be proved directly like in the case of polynomial rings [1].

In [10, Lemma 3.6] the following result was proved.

THEOREM 2.3. *Let A be a ring. The following are equivalent:*

(a) *A is X_1-invariant,*

(b) *A is X_n-invariant for all $n \geqslant 1$,*

(c) *A is X_n-invariant for some $n \geqslant 1$.*

We are now in a position to show that the effective types of invariance enjoy a similar property. For the proof we will need a more precise version of Lemma 3.2 from [10].

LEMMA 2.4. *Let X be a free abelian group of rank two generated by $\{x_1, x_2\}$ and let Y be the infinite cyclic group generated by $\{y\}$. If $\varphi: RX \to SY$ is an isomorphism then there exist an R-automorphism $\tau: RX \to SY$ and an S-automorphism $\sigma: S \to SY$ such that $\sigma\varphi\tau(x_2) = y$.*

THEOREM 2.5. *Let A be a ring. The following are equivalent*:

(a) *A is strongly X_1-invariant,*

(b) *A is strongly X_n-invariant for all $n \geqslant 1$,*

(c) *A is strongly X_n-invariant for some $n \geqslant 1$.*

Proof. (a) $\Rightarrow$ (b) We proceed by induction on n. Let $n > 1$ and $\varphi: AX_n \overset{\approx}{\to} BX_n$. Take $u \in U(A)$; then $\varphi(u) = v \cdot \sum e_i x_i + p$. By Theorem 2.1 it is enough to show that $x_i = 1$ for all i. Assume the contrary, e.g., $x_1 \neq 1$. Set $e = e_1$ and $f = \bar{\varphi}^1(e)$. Then φ induces an isomorphism $fAX \to eBX$ and $\varphi(fu) = vex_1 + p'$. In view of Proposition 1.1 we can and will assume that $\varphi(u) = vx + p$, $x \neq 1$.

Pick a basis $\{y_1, ..., y_n\}$ for X_n so that $x = y_1^k$, $k > 0$. Put $R = A\langle y_1, ..., y_{n-2}\rangle$, $S = B\langle y_1, ..., y_{n-1}\rangle$. Applying Lemma 2.4 we can additionally assume that $\varphi(y_n) = y_n$. Therefore

$$\varphi((1 - y_n) AX_n) = (1 - y_n) BX_n$$

and φ factors to $\bar{\varphi}: A\langle y_1, ..., y_{n-1}\rangle \overset{\approx}{\to} B\langle y_1, ..., y_{n-1}\rangle$ with

$$\bar{\varphi}(u) = vx + p'.$$

By the induction hypothesis A is strongly X_{n-1}-invariant and so by Theorem 2.1 we obtain that $\bar{\varphi}(u) \in U(B) + P(BX_{n-1})$. This is a contradiction.

(b) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$ (a) If $\varphi: AX_1 \to BX_1$ is an isomorphism then for any $n \geqslant 1$, φ induces an isomorphism $\varphi_n: AX_n \to BX_n$ extended by identity on the new generators. This remark together with Theorem 2.1 completes the proof. ∎

THEOREM 2.6. *Let A be a ring. The following are equivalent*:

(a) *A is totally X_1-invariant,*

(b) *A is totally X_n-invariant for all $n \geqslant 1$,*

(c) *A is totally X_n-invariant for some $n \geqslant 1$.*

Proof. (a) $\Rightarrow$ (b). We proceed by induction. Let $n > 1$ and $\varphi: AX_n \overset{\approx}{\to} BX_n$. By assumption and by Theorem 2.5 we know that A is strongly X_n-invariant. Therefore we can assume that φ is an X_n-isomorphism. Take an element $a \in A$. Write $\varphi(a) = \sum b_i x_i$ with x_i all different. Assume that some x_i, say x_1, is different from 1. It is easy to find an epimorphism $\alpha: X_n \to X_{n-1}$ such that $\alpha(x_1) \neq 1$ and $\alpha(x_i) \neq \alpha(x_1)$ for $i > 1$. Obviously α induces epimorphisms

$$\alpha_1: AX_n \to AX_{n-1} \qquad \text{and} \qquad \alpha_2: BX_n \to BX_{n-1}.$$

The isomorphism φ, being an X_n-homomorphism, factors to

$$\bar{\varphi}: AX_{n-1} \to BX_{n-1}.$$

Now $\bar{\varphi}(a) = \sum b_i \, \alpha(x_i) \notin B$. This contradicts the induction hypothesis. Thus $\varphi(a) \in B$, as desired.

The implications (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) can be proved like in the proof of Theorem 2.5. ∎

3. Intrinsic Characterisations

From the previous section it follows that when studying free abelian groups of finite rank we might restrict our attention to the infinite cyclic group. In this case the definitions of the strong and total X-invariance can be interpreted as in [13] in the following way. The ring A is strongly X_1-invariant iff $A[x, x^{-1}] = B[y, y^{-1}]$ implies $A[x, x^{-1}] = B[x, x^{-1}]$. Similarly, the ring A is totally X_1-invariant iff $A[x, x^{-1}] = B[y, y^{-1}]$ implies $A = B$. Here the letters x, y stand for variables.

It comes out that in the above definition of strong invariance we can additionally assume that B is isomorphic to A. We are going to prove the last statement using the terminology of the previous section. In particular we continue to assume that X is free of finite rank. In this way we arrive to an intristic characterisation of the effective invariance types for such groups.

Theorem 3.1. *Let A be a ring, $X = \langle x \rangle$: an infinite cyclic group. The following are equivalent*:

(a) *A is strongly X-invariant,*

(b) *for any automorphism $\varphi: AX \to AX$ we have $\varphi(U(A)) \subseteq U(A) + P(AX)$,*

(c) $\mathrm{Aut}(AX) = \mathrm{Aut}_X(AX) \times \mathrm{Aut}_A(AX)$ *as sets.*

Here $\mathrm{Aut}_X(AX)$ (*resp* $\mathrm{Aut}_A(AX)$) stands for the set of X-automorphisms (*resp. A-automorphisms*) *of AX.*

Proof. (a) $\Rightarrow$ (c) Directly from the definition.

(c) $\Rightarrow$ (b) Take $\varphi \in \mathrm{Aut}(AX)$. By assumption, $\varphi = \varphi_1 \circ \varphi_2$ where $\varphi_1 \in \mathrm{Aut}_X(AX)$ $\varphi_2 \in \mathrm{Aut}_A(AX)$. Obviously $\varphi(U(A)) = \varphi_1(U(A))$. Thus we can assume that φ is an X-automorphism.

Take $u \in U(A)$. Write $\varphi(u) = v \cdot \sum_{i=k}^{l} e_i x^i + p$ with idempotents e_k, $e_l \neq 0$.

Assume that $l > 0$. Define an A-automorphism $\tau: AX \to AX$ by setting $\tau(x) = ux$. Then

$$\varphi\tau(x) = \varphi(ux) = v \sum_{i=k}^{l} e_i x^{i+1} + p'.$$

By assumption, $(\varphi\tau)^{-1}$ has a decomposition: $(\varphi\tau)^{-1} = \psi_1^{-1} \circ \psi_2^{-1}$ where $\psi_1 \in \mathrm{Aut}_X(AX)$, $\psi_2 \in \mathrm{Aut}_A(AX)$. That is, $\varphi\tau = \psi_2 \circ \psi_1$. Let us compare the values of both sides at x.

$$v \cdot \sum_{i=k}^{l} e_i x^{i+1} + p' = \varphi\tau(x) = (\psi_2 \circ \psi_1)(x) = \psi_2(x) \qquad \text{and} \qquad \psi_2 \in \mathrm{Aut}_A(AX).$$

Let us consider the restriction σ of ψ_2 to $e_l AX$. Then σ is an $e_l A$-automorphism such that $\sigma(x) = e_1 v x^{l+1} + p''$. Therefore σ is elementary but in the same time h_σ is not an automorphism. This contradicts Theorem 1.4. It must be then $l \leqslant 0$. Similarly we can verify that $k \geqslant 0$, and so $\varphi(u) = v + p \in U(A) + P(AX)$.

(b) $\Rightarrow$ (a) Let $\varphi: AX \to BX$ be an isomorphism. In [12, Theorem IV.4.5] and in [10, Sect. 2] it was proved that A and B must be subisomorphic. More precisely, there exists an overring C of B, isomorphic to A, such that B is a direct summand of C as a B-module. Moreover, there is an isomorphism $\psi: AX \to CX$ such that $\psi|_A = \varphi|_A$. Let $\beta: C \to A$ be an isomorphism. Then β induces an X-isomorphism $\tau: CX \to AX$. The composition $\tau\psi$ is an automorphism of AX, to which we apply condition (b). Consequently, if $u \in U(A)$ then $\tau\psi(u) \in U(A) + P(AX)$. Therefore $\psi(u) \in U(C) + P(CX)$. However, $\psi(u) = \varphi(u) \in U(BX)$. Because B is a direct summand of C, we get $\psi(u) \in U(B) + P(BX)$. By Theorem 2.1 we obtain (a). ∎

Remark 3.2. By the above theorem and the apparent possibility of extending automorphisms from AX_1 to AX_n one can easily prove Theorem 3.1 for an arbitrary X_n with $n \geqslant 1$.

In the case of the total invariance we have the following analogue of the previous theorem.

THEOREM 3.3. *For any ring A and $n \geqslant 1$ the following conditions are equivalent*:

(a) *A is totally X_n-invariant,*

(b) *For any $\varphi \in \mathrm{Aut}(AX)$ holds $\varphi(A) = A$.*

Proof. (a) $\Rightarrow$ (b) Clear from definition.

(b)$\Rightarrow$(a) Remark 3.2 implies that A is strongly X_n-invariant. Let $\varphi: AX_n \to BX_n$ be an isomorphism. As A is isomorphic to B, we can choose an isomorphism $\beta: B \to A$ and extend it to an X-isomorphism $\tau: BX_n \to AX_n$. Now, $\tau\varphi$ is an automorphism of AX_n. By assumption, $\tau\varphi(A) = A$, so $\varphi(A) = B$, as τ was induced by a map of coefficients. ▮

An analogue of (a)$\Leftrightarrow$(c) from Theorem 3.1 seems to be of more complicated nature.

PROPOSITION 3.4. *Let $n \geqslant 1$ and let A be a ring such that for any $b \in AX_n \setminus A$ there exists an automorphism $\sigma \in \mathrm{Aut}_A(AX_n)$ such that $\sigma(b) \neq b$. Then the following conditions are equivalent:*

(a) *A is totally X_n-invariant,*

(c) *$\mathrm{Aut}\, AX$ is a semidirect product of its normal subgroup $\mathrm{Aut}_A(AX)$ with $\mathrm{Aut}(A)$.*

Proof. (a)$\Rightarrow$(c) If $\varphi \in \mathrm{Aut}(AX_n)$ then by assumption $\varphi|_A \in \mathrm{Aut}(A)$. The restriction map is really a group homomorphism with kernel $\mathrm{Aut}_A(AX)$. But the group $\mathrm{Aut}(A)$ can be treated as a subgroup of $\mathrm{Aut}_{X_n}(AX_n)$ in a natural way.

(c)$\Rightarrow$(a) Let φ be an automorphism of AX. By Theorem 3.3 it is sufficient to show that $\varphi(A) \subseteq A$. Take $a \in A$ such that $\varphi(a) \in AX_n \setminus A$. By assumption we have an A-automorphism σ of AX_n such that $\sigma\varphi(a) \neq \varphi(a)$. By condition (c), $\sigma\varphi = \varphi\tau$ where $\tau \in \mathrm{Aut}(A)$. Thus $\sigma\varphi(a) = \varphi\tau(a) = \varphi(a)$: a contradiction. ▮

It is easy to see that the assumptions of the last proposition are fulfilled in particular when either $n \geqslant 2$ or A is an algebra over an infinite field. On the other hand, if A is a finite field then conditions (a) and (c) are still equivalent, without the extra assumption on A being satisfied. It would be worthwhile to clarify the situation completely.

4. GROUPS OF FINITE RANK

In this section we will assume that Y is a group of finite rank, although not necessarily free. By X_n we continue to denote a free abelian group of rank n. For most groups Y there exist rings of all three kinds of invariance. It is related to the cancelation properties in abelian groups, see [2; 3; 5, Sect. 90].

THEOREM 4.1. *Let A be a von Neumann regular ring. Then A is totally Y-invariant.*

Proof. Let $\varphi: AY \to BY$ be an isomorphism. By Lemma 1.3 we can assume that φ is elementary. Thus h_φ can be considered.

Let I denote a minimal prime ideal in A. Then IY is a minimal prime ideal in AY and hence $\varphi(IY) = JY$ for some minimal prime ideal J in B. After a factorisation we obtain $\bar{\varphi}: (A/I)\,Y \to (B/J)\,Y$. Clearly, $h_{\bar{\varphi}} = h_\varphi$. The regularity of A implies that A/I is a field. By the triviality of units we know that $\bar{\varphi}(A/I) \subseteq B/J$. In particular h_φ is a surjection. As Y is of finite rank, h_φ must be an automorphism.

Because A (and hence B) are reduced so Theorem 1.4 implies that the B-endomorphism $\sigma: BY \to BY$ given by $\sigma(y) = \varphi(y)$ is an automorphism. This easily implies that A is strongly Y-invariant. Thus we can also assume that φ is an Y-isomorphism.

We observed earlier that for all minimal prime ideals $I \subseteq A$ holds $\bar{\varphi}(A/I) \subseteq B/J$ for a suitable ideal $J \subseteq B$. This gives $\varphi(A) \subseteq B$. Therefore $BY = \varphi(AY) = \varphi(A)\,Y$. Thus $\varphi(A) = B$. ∎

THEOREM 4.2. *If A is a von Neumann regular ring then the polynomial ring $A[t]$ is strongly Y-invariant but it is not totally Y-invariant.*

Proof. Let us note that each minimal prime ideal in $A[t]$ is of the form $I[t]$ where I is a minimal prime ideal in A. Moreover, $U(A[t]) = U(A)$, as A is reduced. Thus the first part of Theorem 4.2 can be proved exactly like in the proof of Theorem 4.1.

Now, if $y \in Y \setminus \{1\}$ then we can consider a $A[Y]$-automorphism φ of $(A[t])\,Y$ such that $\varphi(t) = ty$. Clearly, $\varphi(t) \notin A[t]$. Hence $A[t]$ is not totally Y-invariant. ∎

In the above theorem one could use in place of $A[t]$ a polynomial ring on any (nonempty) set of variables. To proceed further we will need

LEMMA 4.3. *Let K be a field, B a ring, and G a torsion-free abelian group. If $(KG)\,Y \approx BY$ then there exists a group H such that $B \approx KH$ and $G \times Y \approx H \times Y$.*

Proof. Let $\varphi: (KG)\,Y \to BY$ be the given isomorphism. As the coefficient rings are domains, the groups of units are trivial and so $\varphi: U(K) \times G \times Y \xrightarrow{\approx} U(B) \times Y$. Because $\varphi(K) \subseteq B$, the map φ factors to $G \times Y \approx H \times Y$ where $H = U(B)/\varphi(U(K))$. ∎

A typical example of an X_1-invariant ring which is not strongly X_1-invariant is KX_1 with K, a field. In our setting we obtain the following proposition.

PROPOSITION 4.4. *Let K be a field. Then KX_1 is Y-invariant. Moreover, KX_1 is not strongly Y-invariant iff Y admits a projection onto an infinite cyclic group.*

Proof. Let $\varphi\colon (KX_1)Y \to BY$ be an isomorphism. Lemma 4.3 provides us with a group H such that $B \approx KH$ and $X_1 \times Y \approx H \times Y$. It is easy to see that $Y \approx X_r \times G$ where G admits no projection onto X_1. Therefore $X_{r+1} \times G \approx H \times X_r \times G \approx X_r \times H \times G$.

As G is of finite rank, there exists an epimorphism $H \times G \to X_1$. By the choice of G it follows that there is also an epimorphism $H \to X_1$. As H is of rank one, it is isomorphic to X_1. Thus $B = KH \approx KX_1$, so KX_1 is Y-invariant.

For the second part we again use the decomposition $Y = X_r \times G$. If $r > 0$ then we can consider an automorphism α of $X_1 \times X_r \times G$ which permutes X_1 with one of its isomorphic copies in X_r. We extend α to an K-automorphism $\sigma\colon (KX_1)Y \to (KX_1)Y$ which is elementary. It is clear that $h_\sigma = \alpha|_Y$ is not an automorphism of Y. Thus, by Theorem 1.4 one can verify that KX_1 is not strongly Y-invariant.

Consider now the case $r = 0$, that is $Y = G$. Let $\varphi\colon (KX_1)Y \to BY$ be an isomorphism. For $g \in Y$ we put

$$\varphi(g) = \alpha(g) \cdot \beta(g) \qquad \text{with} \quad \alpha(g) \in U(B), \beta(g) \in Y.$$

We can also assume that $\varphi^{-1}(g) = \gamma(g) \cdot \delta(g) \cdot \eta(g)$, where $\gamma(g) \in U(K)$, $\delta(g) \in X_1, \eta(g) \in Y$. All maps defined above are group homomorphisms. By the choice of G we know that δ is trivial. For arbitrary $g \in Y$ we have

$$g = \varphi\varphi^{-1}(g) = \varphi(\gamma(g) \cdot \eta(g)) = \varphi(\gamma(g)) \cdot \alpha(\eta(g)) \cdot \beta(\eta(g)).$$

As $\varphi(K) \subseteq B$, the first two terms in the above product lie in B. Thus $g = \beta(\eta(g))$ and so β is an epimorphism. Therefore β is an automorphism of Y, by the finiteness of its rank. It follows that the B-endomorphism $\tau\colon BY \to BY$ given by $\tau(g) = \varphi(g) = \alpha(g) \cdot \beta(g)$ is an automorphism. Now we can set the map σ from the definition of the strong invariance to be equal to τ^{-1}. ∎

Again, instead of X_1 we could use any free abelian group.

Remark 4.5. Another possibility of producing examples of rings which are Y-invariant, but not strongly Y-invariant is to take KY itself, K, a field. Such a ring is for sure not strongly Y-invariant. However, it is not always Y-invariant. By Lemma 4.3 and Corollary 1.5 one can easily see that KY is Y-invariant iff $Y \times Y \approx Y \times H$ implies $Y \approx H$.

The above-mentioned cancelation property is satisfied for example when Y is divisible. However, in this case we know a lot more.

THEOREM 4.6. *Let Y be a divisible group. Then each reduced ring is Y-invariant.*

Proof. Assume Y is of rank one. Let A be a reduced ring and $\varphi: AY \to BY$, an isomorphism. By Lemma 1.3 we can assume that φ and φ^{-1} are elementary. Because A (and hence B) are reduced, we can write for $y \in Y$,

$$\varphi(y) = \alpha(y)\,\beta(y), \qquad \varphi^{-1}(y) = \gamma(y)\,\delta(y)$$

with $\alpha(y) \in U(B), \gamma(y) \in U(A)$, $\beta(y)$, $\delta(y) \in Y$. Obviously the maps α, β, γ, δ are group homomorphisms.

If β is nontrivial then, by the choice of Y, β is an automorphism. In this case, like at the end of the proof of Proposition 4.4, we can construct a B-automorphism $\sigma: BY \to BY$ such that $\sigma\varphi: AY \to BY$ is an Y-isomorphism and so $A \approx B$.

If δ is nontrivial a symmetric argument leads to the same conclusion. Finally, assume $\beta(y) = \delta(y) = 1$ for all $y \in Y$. Let τ be an A-automorphism of AY given by $\tau(y) = \gamma(y) \cdot y$ (see Theorem 1.4). Now $\varphi\tau(y) = \varphi(\gamma(y)\, y) = \varphi(\gamma(y)\,\delta(y)) \cdot \varphi(y) = \varphi(\varphi^{-1}(y)) \cdot \varphi(y) = \alpha(y) \cdot y$. Thus $\varphi\tau$ is elementary and $h_{\varphi\tau}$ is an isomorphism. The first part of the proof guarantees $A \approx B$. This ends the proof in the case of rank one.

In general, the group ring of an arbitrary torsion-free group G with reduced coefficients is itself reduced. This makes possible an induction on rank Y, which ends the proof. ∎

It is well known that there exist domains which are not X_1-invariant. (see [8, 13]). If A is such a domain and $Y = X_1 \times G$ then A is not Y-invariant. Another way of constructing rings which are not Y-invariant is to find torsion-free abelian groups G, H such that $G \times Y \approx H \times Y$ but $G \not\approx H$ and to apply Corollary 1.5.

5. Remarks and Comments

An analogue of Proposition 1.1 for X-invariance is valid if A is a finite product of rings without nontrivial idempotents. The author does not know whether it is true for an arbitrary ring A.

Corollary 1.2 is an immediate consequence of Proposition 1.1. For the effective types of invariance, in the context of free abelian groups of finite rank, this corollary can be strenghtened to the following.

Theorem 5.1. *Let X be a free abelian group of finite rank. Let $\{A_i\}_{i \in I}$ be a family of rings. Then $\prod_{i \in I} A_i$ is strongly (totally) X-invariant iff A_i is strongly (totally) X-invariant for all $i \in I$.*

Proof. We give the proof for the case of strong invariance only. Let

$A=\prod_{i\in I} A_i$. If A is strongly X-invariant then so are all A_i by Proposition 1.1.

Assume now that all A_i are strongly X-invariant but that A is not. By Theorem 2.1 there must exist a ring B, an isomorphism $\varphi: AX \to BX$ and $u \in U(A)$ such that $\varphi(u) \notin U(B)+P(BX)$. We can write $\varphi(u)=v\sum e_j x_j + p$, as usual. We can assume that $x_1 \neq 1$. Set $e=e_1$ and $f=\bar{\varphi}^1(e)$. Then $f=(f_i)_{i\in I}$. Consider

$$B'=eB \qquad \text{and} \qquad A'=fA=\prod_{i\in I} f_i A_i.$$

The homomorphism φ restricts to an isomorphism $\varphi: A'X \to B'X$. In other words, we can assume that $\varphi(u)=vx+p$ where $x\neq 1$.

Let $i\in I$ be chosen so that $f_i \neq 0$. We put $e_i=\varphi(f_i)\in B$. Then

$$\varphi(A_i)X=\varphi(f_i A)X=e_i BX$$

and the restriction of φ to $A_i X$ is such that $\varphi(f_i u)=e_i vx+p'$ where $v\in U(B)$, $p'\in P(e_i BX)$. Also $\varphi(f_i u)\notin U(e_i B)+P(e_i BX)$. But A_i was supposed to be strongly X-invariant. We got a contradiction with Theorem 2.1. ∎

The basic tool in the proofs in sections two, three and four was Corollary 1.2 as well as Lemma 1.3. The latter lemma does not generalize to groups of infinite rank (see Example 4.9 in 9).

In the results of sections two and three on free groups a special role was played by Theorem 2.1. Again, it cannot be extended to arbitrary groups, even of finite rank. On one hand let us consider any noncyclic group Y of rank one. If we apply Proposition 4.4 we obtain a ring KX_1 which is strongly Y-invariant together with an automorphism φ such that $\varphi(U(KX_1))$ is not contained in $U(KX_1)$. On the other hand we have

EXAMPLE 5.2. Let K be a field of characteristic p. Consider the polynomial ring $K[t_1, t_2,...]$ and its ideal I generated by $\{t_1^p, t_2^p-t_1, t_3^p-t_2,...\}$. Set $A=K[t_1,t_2,...]/I$ with $a_i=t_i+I$. In particular A is a local ring and $A/P(A)$ is a field isomorphic to K. Take Y equal to a torsion-free p-divisible abelian group of rank one. Let $\sigma\in \mathrm{Aut}(AY)$. Reducing the ring AY modulo p the map σ induces an automorphism $\bar{\sigma}$ of KY. Thus we get $\bar{\sigma}(K)=K$. This gives that

$$\sigma(U(A))\subseteq U(A)+P(AY).$$

However, we show that A is not strongly Y-invariant.

Pick $g \in Y \backslash \{1\}$. Then any element $y \in Y$ can be written as $y = g^{r/s}$ for some relatively prime integers r, s. Set $g_0 = g$, $g_{n+1} = g_n^{1/p}$ for $n \geqslant 0$. Now, each element $y \in Y$ can be uniquely written as $y = g_n^{r/s}$ with r, s relatively prime and $p \nmid s$.

Let

$$G = \langle g^{r/s}: (r, s) = 1, p \nmid s \rangle \qquad \text{and} \qquad H = \langle g_1, g_2, \ldots \rangle.$$

It is clear that $G \cdot H = Y$ and $G \cap H = \langle g \rangle$.

We define a G-automorphism $\varphi: AY \to AY$ which is K-linear and such that

$$\varphi(g_i) = g_i + a_i g_{i+1}, \qquad \varphi(a_i) = a_i g_{i+1} \qquad \text{for} \quad i \geqslant 1.$$

By the choice of elements a_i it follows that φ is well defined and that φ is really an automorphism of AY. Also

$$\varphi(A(A)) \subseteq U(A) + P(AY).$$

Assume that A is strongly Y-invariant. Then for a suitable A-automorphism σ of AY holds: $\sigma^{-1}(y) = \varphi(y)$. In particular $\sigma^{-1}(g_0) = g_0$ and $\sigma^{-1}(g_i) = g_i + a_i g_{i+1}$ for $i \geqslant 1$. One can calculate now like in [9, Example 6.3] that no A-automorphism can satisfy these relations: there is no element $b \in AY$ satisfying $\sigma^{-1}(b) = g_1$.

The author does not know whether the ring A considered above is Y-invariant. When Y is divisible, this question is interesting in view of Theorem 4.5.

Finally, let us consider the case of groups of infinite rank. It is not possible to extend most of our results to infinitely generated torsion-free abelian groups. Let, say X, be a free abelian group of infinite rank. It is clear that $X \approx X_1 \times X$. Thus, if A is an X-invariant ring then $A \approx AX_1$, so A must be rather pathological; in particular it cannot satisfy any finiteness conditions. However, we will provide an example of such a ring.

EXAMPLE 5.3. Let G be a free abelian group with rank $G >$ rank X. For convenience, let K be a field. We show that $A = KG$ is X-invariant.

Let B be a ring and let $\varphi: AX \to BX$ be an isomorphism. Lemma 4.3 gives us a group H such that $B \approx KH$ and $G \times X \approx H \times X$. Thus H is a free abelian group. A simple calculation on ranks shows that $G \approx H$. Thus $A = KG \approx KH = B$.

As to the other types of invariance, the situation for free groups of infinite rank is even worse.

PROPOSITION 5.4. *Let X be a free abelian group of infinite rank. There are no strongly X-invariant rings.*

Proof. Assume A was such a ring. Let $B = AX_1$. Consider an isomorphism $\alpha\colon X \to X_1 \times X$. It induces an A-isomorphism

$$\varphi\colon \quad AX \to (AX_1)X = BX.$$

From the assumption it follows that there is a B-automorphism σ of BX such that $\sigma\varphi$ is an X-isomorphism. Let $x \in X \backslash \{1\}$ be such that $\alpha(x) \in X_1$. Then $x = \sigma\varphi(x) = \varphi(x) \in B$ which is impossible. ∎

This proposition implies that there are no totally X-invariant rings, as well.

If our group X happens to be "sufficiently nonhomogeneous" then one can produce examples of rings which are strongly and even totally X-invariant.

EXAMPLE 5.5. For each prime p let $X(p)$ be the unique torsion-free abelian group of rank one which is not p-divisible but is q-divisible for all $q \neq p$. Let X be the discrete product (direct sum) of all $X(p)$. Obviously X is of infinite rank. Let K be a field. We show that K is totally X-invariant.

Let $\varphi\colon KX \to BX$ be an isomorphism. From Lemma 4.3 we obtain a group H such that $B = KH$ and $H \times X \approx X$. Let α be a fixed isomorphism $X \times H \to X$. Then for any prime p and $X(p) \subseteq X \times H$ holds $\alpha(X(p)) \subseteq X(p) \subseteq X$ by the characteristic property of $X(p)$. Hence $\alpha(X) \subseteq X$ and it intersects each $X(p)$ in a nontrivial way. On the other hand, $\alpha(X)$ is a direct summand of X. This implies $\alpha(X) = X$ and so $H = \{1\}$, i.e., $B = K$. Finally, $\varphi(K) = B \approx K$.

REFERENCES

1. S. ABHYANKAR, W. HEINZER, AND P. EAKIN., On the uniqueness of the coefficient ring in a polynomial ring, *J. Algebra* **23** (1972), 310–342.
2. D. M. ARNOLD, Strongly homogenous torsion-free abelian groups of finite rank, *Proc. Amer. Math. Soc.* **56** (1976), 67–78.
3. D. M. ARNOLD AND E. L. LADY, Endomorphism rings and direct sums of torsion-free abelian groups, *Trans. Amer. Math. Soc.* **211**(1975), 225–237.
4. D. B. COLEMAN AND E. E. ENOCHS, Isomorphic polynomial rings, *Proc. Amer. Math. Soc.* **27** (1971), 247–252.
5. L. FUCHS, "Infinite Abelian Groups," Vol. II, Academic Press, New York, 1973.
6. G. KARPILOVSKY, "Commutative Group Algebras," Dekker, New York, 1983.
7. J. KREMPA, On semi-group rings, *Bull. Polon. Acad. Sci.* **25** (3) (1977), 225-231.
8. J. KREMPA, "Isomorphic group rings with nonisomorphic coefficients, *Proc. Amer. Math. Soc.* **83** (1981).

9. J. KREMPA, "Homomorphisms of Group Rings," Univ. Alg. and Appl., Banach Center Publ., Vol. 9, PWN, Warsaw, 1982.
10. J. KREMPA, Isomorphic group rings of free abelian groups, *Canad. J. Math.* **34**(1) (1982), 8–16.
11. D. C. LANTZ, R-automorphisms of RG for G abelian torsion-free, *Proc. Amer. Math. Soc.* **61** (1976), 1–6.
12. S. K. SEHGAL, "Topics in Group Rings," Dekker, New York, 1978.
13. K. YOSHIDA, On the coefficient ring of a torus extension, *Osaka J. Math.* **17** (1980), 769–782.